MISB MOTION IMAGERY STANDARDS BOARD **STANDARD** **Natural Representation of Orbital State Vectors**	**MISB ST 1504** **27 October 2016**

1 Scope

The first step in the photogrammetric workflow is to establish the position of the collecting sensor. Collectors in Earth orbit have velocity components on the order of kilometers per second, so interpolation of simple positions and velocities is not very accurate given the potential differences in frame rates and position updates. Fortunately, this is a well understood problem; for the simple two-body problem it is possible to cast the six constants of integration of the equations of motion in such a way that five do not vary with time. Even in the more complex case of the real world, these orbital elements vary only slowly with time, allowing for more precise interpolations of position.

The purpose of this Standard (ST) is to communicate effectively, efficiently, and with the greatest resulting accuracy the information (the state vector) necessary to determine the position of an orbiting Motion Imagery collector. Note that this ST is not intended to present the force model underpinning the equations of motion; nor does it present the information necessary to construct the state transition matrix for propagating an orbit. It does, however, establish how to represent kinematics of orbital motion, if not the dynamics.

This ST also supports the NGA responsibilities defined in [1], below, specifically "Coordinating with the Intelligence Community to ensure consistency and avoid errors in the translation between the Celestial Reference Frame (CRF) and Terrestrial Reference Frame (TRF)."

While it is sometimes useful to represent the state vector of a satellite in Cartesian coordinates, this capability already exists in other MISB documents such as ST 0801 [2]. For that reason, such a representation is beyond the scope of this Standard.

2 References

[1] Department of Defense Instruction 4650.09 Celestial Reference Frame (CRF) Management., 15 04 2015.

[2] MISB ST 0801.5 Photogrammetry Metadata Set for Digital Motion Imagery, Feb 2014.

[3] U. S. N. Observatory, Astronomical Almanac for the Year 2015, USNO/Nautical Almanac Office, 2014.

[4] F. Casella et al, High-Accuracy Orbital Dynamics Simulation through Keplerian and Equinoctial Parameters., Modelica, 2008, pp. 505-514.

[5] MISB MISP-2017.1: Motion Imagery Handbook, Oct 2016.

[6] MISB ST 0603.4 MISP Time System and Timestamps, Feb 2016.

[7] MISB ST 1010.3 Generalized Standard Deviation and Correlation Coefficient Metadata, Oct 2016.

[8] P. K. Seidelmann, Explanatory Supplement to the Astronomical Almanac, Sausalito: University Science Books, 2006.

[9] R. A. Broucke and P. J. Cefola, "On the Equinoctial Orbit Elements," *Celestial Mechanics,* vol. 5, pp. 303-310, 1972.

[10] V. G. Szebehely, Adventures in Celestial Mechanics: A First Course in the Theory of Orbits, Austin: University of Texas Press, 1989.

3 Symbols and Definitions

The symbols and terms defined herein are limited to the scope of this document, without regard for potential use in or conflict with other MISB documents.

A dotted symbol should be interpreted as the derivative with respect to time of the base quantity.

3.1 Symbols

Three-dimensional Euclidean space ($\mathbb{R}^3$**)** is assumed. All operators take on their $\mathbb{R}^3$ definitions.

All usages are consistent with references [1] and [3].

3.1.1 Basics

$\hat{\imath}$ First component of an orthonormal spanning set in a right handed $\mathbb{R}^3$

$\hat{\jmath}$ Second component of an orthonormal spanning set in a right handed $\mathbb{R}^3$

$\hat{k}$ Third component of an orthonormal spanning set in a right handed $\mathbb{R}^3$

x Scalar value specifying the magnitude of a vector in the x direction

y Scalar value specifying the magnitude of a vector in the y direction

z Scalar value specifying the magnitude of a vector in the z direction

$\vec{r}$ Position Vector $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

$\vec{v}$ Velocity Vector $\vec{v} = (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$

♈ First Point of Aries

3.1.2 Ellipses and Keplerian Elements

a Semi-Major Axis

b Semi-Minor Axis

e Eccentricity

i Inclination

ω Argument of Periapsis

Ω Longitude of the Ascending Node

3.1.3 Anomalies

M Mean Anomaly

M_0 Mean Anomaly at Epoch

E Eccentric Anomaly

ν or f True Anomaly

τ_0 Mean Anomaly Epoch

3.1.4 Non-singular Equinoctial Elements

The expressions used to define these elements are for an object in a direct orbit. The expressions must be altered slightly for a retrograde orbit [4]. [An object (e.g., a satellite) in a direct orbit orbits in the same direction as the rotation of its primary (e.g., the Earth). An object in a retrograde orbit orbits in the opposite direction.]

$$h = e\sin(\omega + \Omega)$$
$$k = e\cos(\omega + \Omega)$$
$$\lambda = M + \omega + \Omega$$
$$p = tan(\frac{i}{2})\, sin\,\Omega$$
$$q = tan(\frac{i}{2})\, cos\,\Omega$$

3.1.5 Auxiliary Parameters

μ $G(M_{Earth} + M_{Satellite})$

n Mean Motion $= \sqrt{\frac{\mu}{a^3}}$

$\vec{e}$ Eccentricity Vector

$\vec{n}$ Node Vector

p *Semilatus Rectum*

$\vec{h}$ Specific Angular Momentum Vector

ζ Specific Energy

3.1.6 Units

°/s degrees/second

m^3/s^2 meters^3/second^2

m	meters
J/kg	joules/kilogram

3.2 Definitions

Apoapsis	General term for the point in an orbit farthest from the central body
Apogee	Term for the point in an orbit farthest from the Earth
Ascending Node	The point in space where an orbiting object passes "up" from the Southern Hemisphere to the Northern Hemisphere through a reference plane, such as, the Equator of the Earth or the plane of the Ecliptic. The Vernal Equinox is the time of this event.
Descending Node	The point in space where an orbiting object passes "down" from the Northern Hemisphere to the Southern Hemisphere through a reference plane, such as, the Equator of the Earth or the plane of the Ecliptic. The Autumnal Equinox is the time of this event.
Eccentric Anomaly	The eccentric anomaly is an angle *E* measured from the center of an auxiliary circle inscribed on an ellipse with radius equal to the semi-major axis. The angle is measured from the line connecting the center of the circle to the occupied focus of the ellipse to the line from the center of the circle to the point defined by the extension of a chord perpendicular to the semi-major axis of the ellipse through the position of the orbiting body onto the auxiliary circle. See Figure 1.

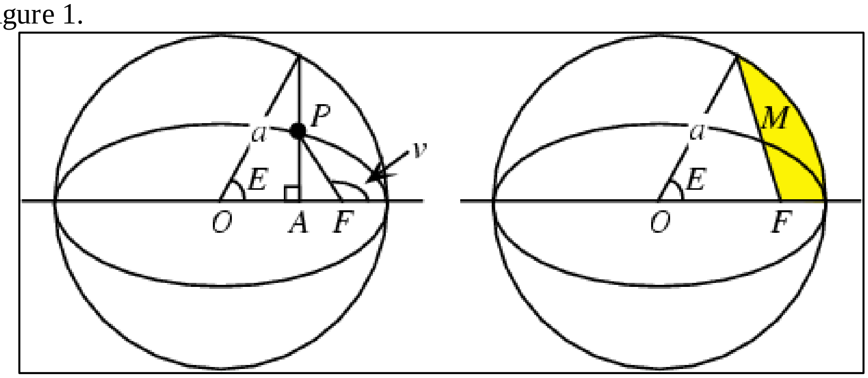

Figure 1: Eccentric, Mean, and True Anomaly

ECEF	Earth-centered, Earth-fixed, a non-inertial frame of reference, which rotates to remain fixed with respect to the surface of the Earth, customarily with the origin at the center of mass of the Earth, the X-axis through the equator at the Greenwich meridian, the Z-axis along the Earth's rotational axis pointing North, and the Y-axis chosen to complete a right-handed, orthogonal system.
ECI	Earth-centered Inertial, an inertial frame of reference, which does not rotate as the Earth rotates, customarily with the Vernal Equinox (the ascending node of the ecliptic) as the X-axis, the Z-axis along the Earth's rotational axis pointing North, and the Y-axis chosen to complete a right-handed orthogonal system.
Ecliptic	The plane of the Earth's orbit around the Sun projected in all directions. The plane of the Ecliptic intersects the celestial sphere (an imaginary sphere of arbitrarily large radius, concentric with Earth) along a great circle, the same

	circle on which the Sun seems to move as the Earth orbits it. The intersections of the Ecliptic and the Equator on the celestial sphere correspond to the times of the Vernal and the Autumnal Equinoxes, where the Sun seems to cross the celestial Equator.
First Point of Aries	Also known as the Cusp of Aries or the Vernal Equinox. The point on the celestial sphere where the ecliptic passes up through the equator. Used as the origin for the x direction in Earth Centered Inertial Frames of Reference.
Julian Day	Cumulative day count from Noon November 24, 4714 BC (Gregorian Calendar). Noon January 1, 2000, was the start of Julian Day 2,451,545.
Kepler's Equation	$M = E - e \sin E$ in its most common formulation. Solving for E is best done numerically. See Figure 1.
Line of Nodes	The line between the Ascending and Descending Nodes
Mean Anomaly	The angle M measured at the occupied focus between the periapsis and the point on the auxiliary circle defined by the extension of a chord perpendicular to the semi-major axis through the position of the body on the orbit to the auxiliary circle. See Figure 1.
Mean Anomaly Epoch	The reference time for which the Mean Anomaly at Epoch was correct
Modified Julian Day	Julian Day − 2400000.5 (smaller number, start of day is midnight)
Periapsis	General term for the point in an orbit closest to the central body.
Perigee	Term for the point in an orbit closest to the Earth
Semilatus Rectum	Half the length of the chord perpendicular to the major axis of an ellipse passing through a focus
Specific	Per unit mass
ST	Standard
State Vector	A vector containing all the variables required to characterize a system at a given time
True Anomaly	The angle ν between the periapsis and the orbiting body as measured from the occupied focus of the ellipse formed by the orbit. See Figure 1.

4 Revision History

Revision	Date	Summary of Changes
ST 1504	10/27/2016	Initial release

5 Frames of Reference

This section is intended as an overview of the differences between inertial and non-inertial frames. FK5 (J2000) and True-of-Date are inertial frames. WGS-84 is a non-inertial frame.

5.1 Inertial vs. Non-Inertial Systems

An inertial system or reference frame is one in which Newton's Laws of Motion hold. This means that in the absence of an outside force, an object will undergo no acceleration. In general, if the reference frame undergoes no accelerations, it will be an inertial frame. In contrast, rotating frames of reference are non-inertial. The surface of the earth is a rotating (hence non-inertial) frame of reference.

5.2 Inertial Frames

To solve for the motion of a satellite around the Earth, an inertial frame simplifies the equations. The question is how best to do this? Inertial Cartesian coordinates (for satellite problems) are usually defined with the Vernal Equinox (the ascending node of the ecliptic) as the X-axis, the rotational North Pole as the Z-axis, and the Y-axis chosen so as to create a right-handed orthogonal system. This is done because, from a viewpoint on the ground, the Vernal Equinox is a fixed point against the unmoving backdrop of the stars. Unfortunately, the equinox and the pole move with time, so a particular epoch must be chosen for which these directions are accurate. These then will become the fixed points used to define an inertial frame of reference; in particular, an Earth-centered Inertial (ECI) frame of reference.

The J2000 system is such a coordinate system, with the X- and Z-axes chosen as on 1 January A. D. 2000 at exactly 1200 Greenwich Time. B1950 is an older and less accurate inertial frame no longer in wide use. Use of B1950 is not supported by this ST. Presumably, at some point during the next 25 years, a new inertial reference frame with a newer epoch will be defined. While these frames are useful, a frame of reference based on the rotational pole and equinox at some other time is desirable. Precession will move the location of the pole and the plane of the equator with time. This yields, for any given time, new directions for the X- and Z-axes. This new frame of reference is called the Mean-of-Date frame, and one may move from the J2000 inertial coordinate system to a Mean-of-Date coordinate system via a rotation matrix. Rotation matrix details are provided in Section 8.1.

5.3 Non-Inertial Frames

Why cast a satellite problem in a non-inertial frame? Nearly all problems come down to, "Is the satellite visible from this ground station?" and/or "Can the satellite see this particular spot on the ground?" To answer these questions, a relation between a satellite and the Earth's (rotating) surface is needed.

Body-fixed (here, Earth-centered, Earth-fixed) coordinates for satellite problems are usually defined with the X-axis through the equator at the Greenwich meridian and the Z-axis though the North Pole. Again, the Y-axis is chosen to complete a right-handed, orthogonal system. WGS84 is the primary Earth-centered, Earth-fixed (ECEF) coordinate system for the Department of Defense and Intelligence Community.

6 Basic Orbital Parameters

The shape of a bound orbit (such as a satellite around the Earth) is an ellipse. Two pieces of information are required to define an ellipse, which usually are the semi-major axis, *a*, and the eccentricity, *e*,

$$e = \sqrt{1 - \frac{b^2}{a^2}} \,,$$

where *b* is the semi-minor axis. (Equivalently, the semi-major axis and the flattening

$$flattening = f = \frac{(a-b)}{a},$$

may be used.) The central body will occupy one focus of the ellipse. The periapsis (this is the general term; for a satellite in orbit around the Earth, the term “perigee” is also correct) represents the point in the orbit of the satellite where it is closest to the central body. The point where the satellite is farthest away from the central body is referred to, in general, as the apoapsis, or for a satellite in orbit around the Earth, as “apogee.”

Once the ellipse that describes the path of the satellite is defined, the orientation of this ellipse with respect to the Earth is needed. This requires an agreed upon reference frame -- an inertial frame (the short definition of which is a frame wherein Newton’s Laws of Motion hold). In general, this requires the frame to be non-accelerated. Also desired is to affix this frame to the (effectively) fixed and unmoving background of the stars.

By convention, the coordinate frame used is chosen with the x-y plane coinciding with the equator of the Earth and the z-axis through the north rotational pole as shown in Figure 2. The origin is at the center of the Earth (and the offset of the geometric center of the Earth and the center of mass of the Earth is ignored). The direction of the x-axis is called the First Point of Aries or Vernal Equinox. It points towards the place on the apparent celestial sphere (the fixed and unmoving background of the stars) where the sun crosses from the Southern Hemisphere to the Northern Hemisphere on the first day of spring. The y-axis is chosen to complete a right-handed Cartesian coordinate system. Now, the first point of Aries and the rotational pole are subject to slow change with time, thus a standard reference date (“epoch”) is required along with the directions as they were at that time. The selected standard reference epoch is noon on January 1, 2000 (Greenwich Time), and the reference frame is called J2000.

The inclination, *i*, is the angle between the satellite orbit’s plane and the equator of the Earth. This is also equivalent to the angle between the angular momentum vector of the satellite and the z-axis. The Longitude of the Ascending Node, Ω, is the angle from the x-axis (also sometimes referred to as the Zero Point of Longitude) to the point where the satellite crosses the equator rising (ascending) from the southern hemisphere to the northern hemisphere.

The descending node, where the satellite moves from the Northern Hemisphere to the Southern Hemisphere, is 180° away. The line through the ascending and descending nodes is referred to as the Line of Nodes. The Argument of Perigee, ω, measures the angle between the Ascending Node and the perigee of the orbit of the satellite. Figure 2 illustrates these angles.

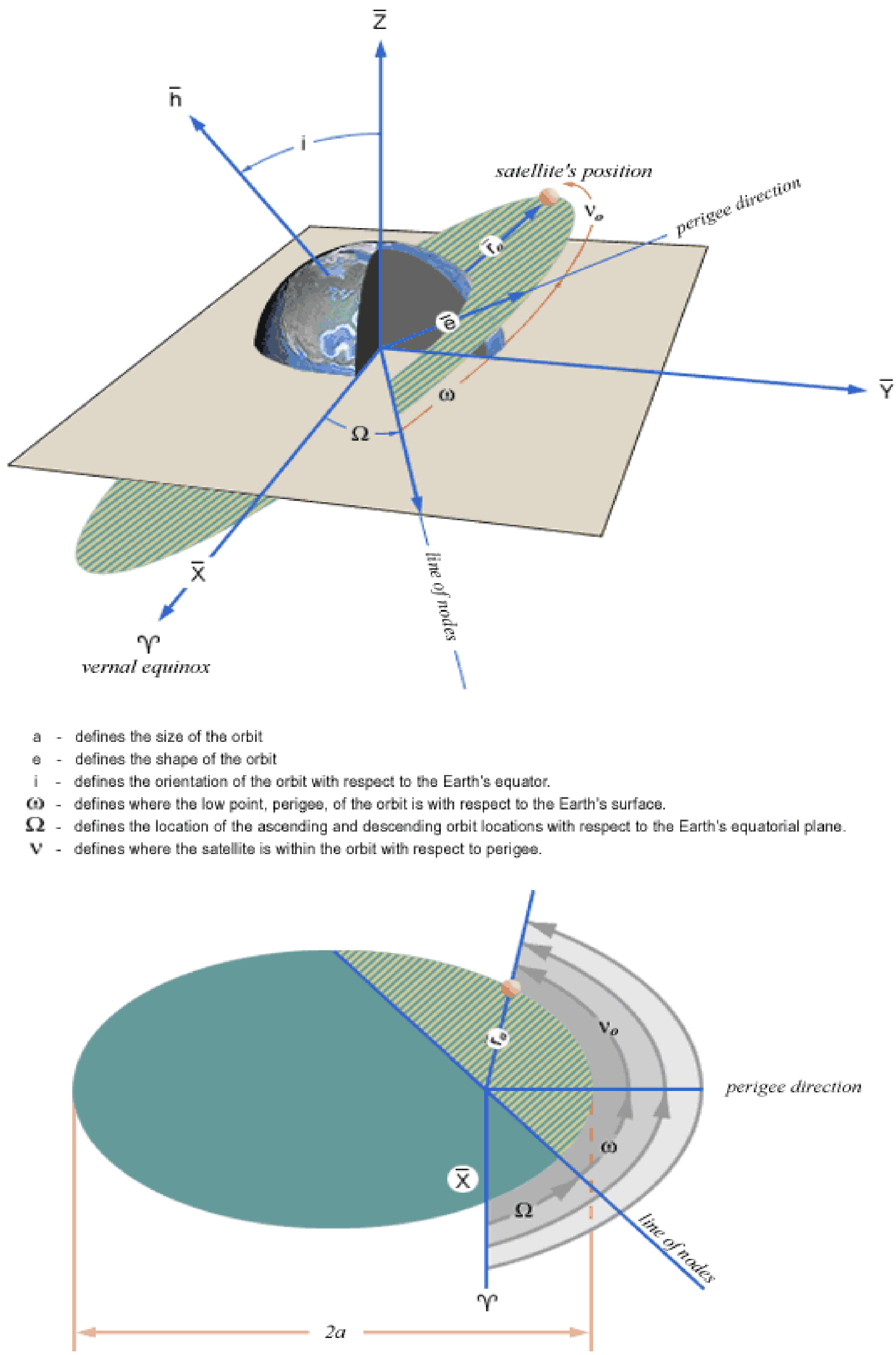

Figure 2: Physical Meaning of Keplerian Elements

All that is needed now is to specify where on this well-defined ellipse embedded in three-dimensional space the satellite is at any given moment. There are several ways of doing this, but the simplest to describe is called the True Anomaly, ν. It is the angle between the position of the satellite and periapsis as measured from the Earth (which, recall, is sitting at one focus of the ellipse, not the geometric center of the ellipse). Figure 2 illustrates the physical meaning and relationships of the orbital elements.

It should be noted that these Keplerian orbital elements can be derived from satellite displacement and velocity vectors. Conversely, satellite displacement and velocity vectors can be derived from the Keplerian orbital elements. Details are presented in Section 8.2 and Section 8.3.

For certain values of inclination (i) and eccentricity (e), direct use of Keplerian elements can lead to singularities. To avoid this problem, the Keplerian elements may be converted to Equinoctial elements, which are non-singular. Details of this transformation are presented in Section 8.4. Section 8.5 and Section 8.6 describe the derivation of Non-Singular Equinoctial orbital elements from satellite displacement and velocity vectors, and the inverse.

7 State Vector Local Set

This section defines a Local Set and five supporting KLV (Key Length Value) Variable Length Packs. The Variable Length Packs, intended to be used within the Local Set, allow for other than double precision floats to be used as necessary.

Two design principles guide this ST. First, minimize the number of KLV keys needed. Instead of defining a key to correspond to every possible orbital element, or any other representation of a state vector, only six generic state vector elements are used. The meaning of a particular state vector element is defined by embedding enumerations for different methods of measurement. This does entail creating controlled vocabularies for other keys that define the meaning and units associated with the state vector elements. This initial version includes the most common and useful controlled vocabulary items consistent with rigor. Future versions of this ST may expand on these items.

Second, use the same units for like quantities (e.g. longitude of the ascending node, argument of perigee, mean anomaly, true anomaly, and eccentric anomaly). While it is theoretically possible, for example, to have a case where the longitude of the ascending node is given as an hour angle (units of hours, minutes, and seconds of *time*), the argument of perigee in decimal degrees of arc, and the anomalies in radians, this is bad practice; the same units should be used for all like quantities.

This ST specifies a single set of units for like quantities. To be conformant with this ST, it is incumbent upon the system that receives the data from the flight article to convert all quantities to the specified units prior to further dissemination of the data. All units are based on the SI system.

The State Vector Local Set is defined in Table 1. The *Tag* column indicates the KLV tag to use when specifying the value in the State Vector Local Set. The *Name* column links the Tag to the purpose of the value as defined in the *Reference* column. The *Key* denotes the KLV dictionary key that is associated with the value. The *Reference* column indicates where more information about the value (i.e. its format and usage rules) is located; the “Sections” refer to this document. *Type* specifies the binary format used when encoding the local set into binary. The *Required* column indicates if the value is Mandatory, Conditional or Optional. If the item is Mandatory the value is required within the Local Set. If the item is Conditional the use of this item is conditional on some other element within the set. If the item is Optional the item is not required to be included in the set. In all cases only one instance of each items is allowed in the set – duplicate items are not allowed.

Table 1: State Vector Local Set

Local Set Key			Name		
06.0E.2B.34.02.0B.01.01.0E.01.03.01.03.00.00.00 (CRC 12461)			State Vector Local Set		
Constituent Elements					
Tag ID	**Name (Symbol)**	**Key**	**Reference**	**Type[1]**	**Required**
1	Precision Time Stamp	06.0E.2B.34.01.01.01.03. 07.02.01.01.01.05.00.00 (CRC 64827)	Section 7.1	uint64	MANDATORY
2	Document Version	06.0E.2B.34.01.01.01.01. 0E.01.02.05.05.00.00.00 (CRC 56368)	Section 7.1	uint	MANDATORY
3	CRC	06.0E.2B.34.01.01.01.01. 0E.01 02.03.5E.00.00.00 (CRC 31377)	Section 7.1	uint16	MANDATORY
4	State Vector Pack	06.0E.2B.34.02.05.01.01. 0E.01.03.01.0C.01.00.00 (CRC 32777)	Section 7.2	dlp	MANDATORY
5	Anomaly Type	06.0E.2B.34.01.01.01.01. 0E.01.01.01.25.00.00.00 (CRC 44184)	Section 7.3	uint	CONDITIONAL
6	Mean Anomaly Epoch	06.0E.2B.34.01.01.01.03. 07.02.01.01.01.05.00.00 (CRC 64827)	Section 7.4	uint	CONDITIONAL
7	State Vector TROC Pack	06.0E.2B.34.02.05.01.01. 0E.01.03.01.0C.02.00.00 (CRC 55641)	Section 7.5	dlp	OPTIONAL
8	Eccentricity Vector Pack ($\vec{e}$)	06.0E.2B.34.02.05.01.01. 0E.01.03.01.0C.03.00.00 (CRC 61033)	Section 7.6	dlp	OPTIONAL
9	Node Vector Pack ($\vec{n}$)	06.0E.2B.34.02.05.01.01. 0E.01.03.01.0C.04.00.00 (CRC 27641)	Section 7.7	dlp	OPTIONAL
10	Specific Angular Momentum Pack	06.0E.2B.34.02.05.01.01. 0E.01.03.01.0C.05.00.00 (CRC 23753)	Section 7.8	dlp	OPTIONAL
11	Mean Motion (n)	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1F.00.00.00 (CRC 59610)	Section 7.9	float	OPTIONAL
12	Mu (μ)	06.0E.2B.34.01.01.01.01. 0E.01.01.01.20.00.00.00 (CRC 4317)	Section 7.9	float	OPTIONAL

Local Set Key			Name		
06.0E.2B.34.02.0B.01.01.0E.01.03.01.03.00.00.00 (CRC 12461)			State Vector Local Set		
Constituent Elements					
Tag ID	**Name (Symbol)**	**Key**	**Reference**	**Type[1]**	**Required**
13	Semilatus Rectum (p)	06.0E.2B.34.01.01.01.01. 0E.01.01.01.21.00.00.00 (CRC 26217)	Section 7.9	float	OPTIONAL
14	Specific Energy (ζ)	06.0E.2B.34.01.01.01.01. 0E.01.01.01.22.00.00.00 (CRC 64949)	Section 7.9	float	OPTIONAL
15	Standard Deviation and Correlation Coefficient FLP	06.0E.2B.34.02.05.01.01. 0E.01.03.03.21.00.00.00 (CRC 64882)	Section 7.10	dlp	OPTIONAL

[1]Note on Lengths: Types with unspecified lengths in the State Vector Local Set are computed by the size of the value. For example, if a uint value is less than 255 then only one byte is needed. See the Motion Imagery Handbook [5] for more information on data types and lengths.

Requirement	
ST 1504-01	Items within MISB ST 1504 State Vector Local Set shall be present at most once in each Local Set.

7.1 MISP Baseline Local Set Parameters

The State Vector Local Set is an application of the MISP Baseline Local Set, so it contains a Precision Time Stamp, Document Version and CRC. All three of these parameters are fully discussed in the Motion Imagery Handbook [5] along with their usage rules.

7.1.1 CRC

The State Vector Local Set is used either standalone within a metadata stream or embedded in another metadata set or pack. When the State Vector Local Set is used standalone, a CRC is included as a validation technique to detect if the State Vector Local Set has been corrupted in some way during transmission. The CRC of the State Vector Local Set is calculated using the technique defined in Motion Imagery Handbook [5].

Requirement(s)	
ST 1504-02	When the State Vector Local Set is not included in another CRC protected metadata set, the State Vector Local Set shall contain a CRC-16-CCITT.
ST 1504-03	The CRC-16-CCITT shall be calculated in accordance with the algorithm in the Motion Imagery Handbook [5].
ST 1504-04	The CRC-16-CCITT shall be the last element in the State Vector Local Set.

7.2 State Vector Element Pack

Table 2 defines the elements in the State Vector Pack. The *Pack Key* and *Name* at the top of the table uniquely identify the pack and are registered with the KLV Dictionary. The *Name* and *Key* columns are the identification of the elements. The *Type* column defines the format of the data for the given element. The *Length* column defines the number of bytes that the elements use in the pack. The *SDCC-FLP Source List* indicates if the element is part of the State Vector Pack's Source List.

Table 2: State Vector Element Pack

Pack Key	Name			
06.0E.2B.34.02.05.01.01.0E.01.03.01.0C.01.00.00 (CRC 32777)	State Vector Element Pack			
Constituent Elements				
Name	**Key**	**Type**	**Length (Bytes)**	**SDCC-FLP Source List**
State Vector Status (SVS)	06.0E.2B.34.01.01.01.01.0E.01.01.01.24.00.00.00 (CRC 55852)	ber-oid	1 (Variable)	No
State Vector Element #1	06.0E.2B.34.01.01.01.01.0E.01.01.01.1A.00.00.00 (CRC 21663)	float	4 or 8 (see (1) below)	Yes
State Vector Element #2	06.0E.2B.34.01.01.01.01.0E.01.01.01.1A.00.00.00 (CRC 21663)	float	4 or 8 (see (1) below)	Yes
State Vector Element #3	06.0E.2B.34.01.01.01.01.0E.01.01.01.1A.00.00.00 (CRC 21663)	float	4 or 8 (see (1) below)	Yes
State Vector Element #4	06.0E.2B.34.01.01.01.01.0E.01.01.01.1A.00.00.00 (CRC 21663)	float	4 or 8 (see (1) below)	Yes
State Vector Element #5	06.0E.2B.34.01.01.01.01.0E.01.01.01.1A.00.00.00 (CRC 21663)	float	4 or 8 (see (1) below)	Yes
State Vector Element #6	06.0E.2B.34.01.01.01.01.0E.01.01.01.1A.00.00.00 (CRC 21663)	float	4 or 8 (see (1) below)	Yes

(1) In this pack the final six elements must use the same length; their individual lengths are computed using the equation below.

$$Length_{Float} = (Length_{Pack} - Length_{SVS})/6$$

All elements in the pack are mandatory. The first element, defines the Reference Frame and State Vector Type, which defines the meaning of the remaining six State Vector Elements in the pack. As mentioned in Section 1, for the simple two-body problem it is possible to specify a frame of reference using six parameters. With this six-element vector, one can represent both inertial and non-inertial frames of reference.

State Vector Status (SVS) is a binary value represented as a BER-OID integer. A BER-OID value is used to enable future expansion, while keeping the State Vector Pack small. The State Vector Status value contains three parts: Flag, Reference Frame, and State Vector Type as shown in Figure 3.

(msb)			Bits				(lsb)
7	6	5	4	3	2	1	0
Flag	Reference Frame			State Vector Type			

Figure 3: State Vector Status Value bit values

The MSB (bit 7 - Flag) is the BER-OID control signal to indicate whether more bytes are needed to represent the value. For this version of ST 1504, bit 7 - Flag is always zero (0) indicating no additional bytes beyond one byte are needed. In future versions when adding more bytes to the Status Value, they will be additional least significant bytes, so the byte shown in Figure 3 will be the most significant byte.

Bits 4-6 signal the type of Reference Frame. Only three values are defined in Table 3; the remaining values are reserved for experimentation (see Section 7.11) or future use. The Reference Frame value indicates which reference frame was used for the measurements in the State Vector Pack.

Table 3: Reference Frame

Value (bits 4-6)	Explanation
0	RESERVED (See Section 7.11)
1	Earth Centered, Earth Fixed as defined by WGS 84 Coordinate Reference Frame. Note that Ω is given in terms of Greenwich Meridian.
2	J2000 Earth Centered Inertial Coordinate System (FK5) Note that Ω is given in terms of First Point of Aries.
3	True of Date Earth Centered Inertial Coordinate System Note that Ω is given in terms of First Point of Aries.
4-7	RESERVED

Bits 0-3 indicate the State Vector Type. The allowed values are listed in Table 4; the remaining values are reserved for experimentation (see Section 7.11) or future use.

Table 4: State Vector Type

Value (bits 0-3)	Explanation
0	RESERVED (See Section 7.11)
1	Keplerian Orbital Elements as defined in Section 6.
2	Non-Singular Equinoctial Elements as defined in Section 8.4
3-15	RESERVED

The meaning and units of the State Vector Elements are determined by State Vector Type and Reference Frame, respectively. Table 5 defines the interpretation of the six State Vector Elements based on the value of the State Vector Type.

Table 5: Interpretation of State Vector Elements

State Vector Type	Interpretation of State Vector in State Vector Element Pack		Units: Measured Relative to Reference Frame (See Table 3)
Keplerian	1	semi-major axis of orbit (a)	Meters
	2	Eccentricity of orbit (e)	Unitless
	3	Inclination of orbit (i)	Decimal Degrees
	4	Argument of perigee (ω)	Decimal Degrees
	5	Longitude of the Ascending Node (Ω)	Decimal Degrees
	6	Anomaly as defined by Anomaly Type (see Section 7.3)	Decimal Degrees
Equinoctial	1	semi-major axis of orbit (a)	Meters
	2	$\mathrm{h} = e\sin(\omega + \Omega)$	Unitless
	3	$\mathrm{k} = e\cos(\omega + \Omega)$	Unitless
	4	$\lambda = \mathrm{M} + \omega + \Omega$	Decimal Degrees
	5	$\mathrm{p} = \tan(\frac{i}{2})\sin\Omega$	Unitless
	6	$\mathrm{q} = \tan(\frac{i}{2})\cos\Omega$	Unitless

Requirement(s)	
ST 1504-05	A State Vector Pack shall contain State Vector Elements of the same length.
ST 1504-06	The Reference Frame value of a MISB ST 1504 State Vector Pack shall be one of the non-RESERVED values defined in MISB ST 1504 Table 3: Reference Frame.
ST 1504-07	The State Vector Type value of a MISB ST 1504 State Vector Pack shall be one of the non-RESERVED values defined in MISB ST 1504 Table 4: State Vector Type.

7.3 Anomaly Type

The values for the Anomaly Type key are given in Table 6.

Table 6: Anomaly Type

Value	Explanation
0	RESERVED (See Section 7.11)
1	Eccentric Anomaly
2	Mean Anomaly
3	Mean Anomaly at Epoch
4	True Anomaly

Requirement(s)	
ST 1504-08	Where the value of the State Vector Type Key is set to one (1) (Keplerian Orbital Elements), the Anomaly type key shall appear exactly once in the instance of the State Vector Local Set.
ST 1504-09	The Anomaly Type value of a MISB ST 1504 State Vector Local Set shall be one of the non-RESERVED values defined in MISB ST 1504 Table 6: Anomaly Type.

7.4 Mean Anomaly Epoch

The value of the Mean Anomaly Epoch is the time at which the Mean Anomaly at Epoch was valid. The time system is defined in MISB ST 0603 [6].

Requirement	
ST 1504-10	Where the value of the Anomaly Type is equal to 3 (Mean Anomaly at Epoch), the Mean Anomaly Epoch shall be present in the State Vector Local Set.

7.5 State Vector Element Time Rate of Change Pack

Table 7 defines the elements in the State Vector Element Time Rate of Change (TROC) Pack. The *Pack Key* and *Name* at the top of the table uniquely identify the pack and are registered in the KLV Dictionary. The *Name* and *Key* columns are the identification of the elements. The *Type* column defines the format of the data for the given element. The *Length* column defines the number of bytes the elements use in the pack. The *SDCC-FLP Source List* indicates if the element is part of the State Vector TROC Pack's Source List. The units of these elements are those of the corresponding State Vector Pack element per second.

This pack is optional, but when used, all elements in the pack are mandatory.

Table 7: State Vector Time Rate of Change Pack

Pack Key		Name		
06.0E.2B.34.02.05.01.01.0E.01.03.01.0C.02.00.00 (CRC 55641)		State Vector Time Rate of Change Pack		
Constituent Elements				
Name	**Key**	**Type**	**Length**	**SDCC-FLP Source List**
State Vector Element #1 TROC	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1B.00.00.00 (CRC 8747)	float	4 or 8 (see (1) below)	Yes
State Vector Element #2 TROC	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1B.00.00.00 (CRC 8747)	float	4 or 8 (see (1) below)	Yes
State Vector Element #3 TROC	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1B.00.00.00 (CRC 8747)	float	4 or 8 (see (1) below)	Yes
State Vector Element #4 TROC	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1B.00.00.00 (CRC 8747)	float	4 or 8 (see (1) below)	Yes
State Vector Element #5 TROC	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1B.00.00.00 (CRC 8747)	float	4 or 8 (see (1) below)	Yes
State Vector Element #6 TROC	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1B.00.00.00 (CRC 8747)	float	4 or 8 (see (1) below)	Yes

(1) In this pack the six elements must use the same length; their individual lengths are computed using the total length of the pack divided by six, as shown in the equation below.

$$Length_{Float} = Length_{Pack}/6$$

The meaning of the elements of these keys is the TROC of the corresponding element of the State Vector Pack.

Requirement	
ST 1504-11	A State Vector TROC Pack shall contain State Vector Elements of the same length.

7.6 Eccentricity Vector Pack

Table 8 defines the elements in the Eccentricity Vector pack. The *Pack Key* and *Name* at the top of the table uniquely identify the pack, and are registered in the KLV Dictionary. The *Name* and *Key* columns are the identification of the elements. The *Type* column defines the format of the data for the given element. The *Length* column defines the number of bytes the elements use in the pack. The *SDCC-FLP Source List* indicates if the element is part of the Eccentricity Vector Pack's Source List. The Eccentricity Vector Packs elements are unit-less.

This pack is optional, but when used, all elements in the pack are mandatory.

The Eccentricity Vector is discussed in Section 6.

Table 8: Eccentricity Vector Pack

Pack Key			**Name**	
06.0E.2B.34.02.05.01.01.0E.01.03.01.0C.03.00.00 (CRC 61033)			Eccentricity Vector Pack	
Constituent Elements				
Name	**Key**	**Type**	**Length**	**SDCC-FLP Source List**
Eccentricity X Component	06.0E.2B.34.01.01.01.01.0E.01.01.01.1C.00.00.00 (CRC 29446)	float	4 or 8 (see (1) below)	No
Eccentricity Y Component	06.0E.2B.34.01.01.01.01.0E.01.01.01.1D.00.00.00 (CRC 1458)	float	4 or 8 (see (1) below)	No
Eccentricity Z Component	06.0E.2B.34.01.01.01.01.0E.01.01.01.1E.00.00.00 (CRC 40558)	float	4 or 8 (see (1) below)	No

(1) In this pack the three elements must use the same length; their individual lengths are computed using the total length of the pack divided by three, as shown in the equation below.

$$Length_{Float} = Length_{Pack}/3$$

Requirement	
ST 1504-12	An Eccentricity Vector Pack shall contain Eccentricity elements of the same length.

7.7 Node Vector Pack

Table 9 defines the elements in the Node Vector Pack. The *Pack Key* and *Name* at the top of the table uniquely identify the pack, and are registered in the KLV Dictionary. The *Name* and *Key* columns are the identification of the elements. The *Type* column defines the format of the data for the given element. The *Length* column defines the number of bytes an element uses in the pack. The *SDCC-FLP Source List* indicates if the element is part of the Node Vector Pack's Source List. The Node Vector Packs elements are the components of a unit vector and do not have measurement units.

This pack is optional, but when used, all elements in the pack are mandatory.

The Node Vector values are discussed in Section 6.

Table 9: Node Vector Pack

Pack Key				Name
06.0E.2B.34.02.05.01.01.0E.01.03.01.0C.04.00.00 (CRC 27641)				Node Vector Pack
Constituent Elements				
Name	**Key**	**Type**	**Length**	**SDCC-FLP Source List**
Unit Vector X Component	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1C.00.00.00 (CRC 29446)	float	4 or 8 (see (1) below)	No
Unit Vector Y Component	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1D.00.00.00 (CRC 1458)	float	4 or 8 (see (1) below)	No
Unit Vector Z Component	06.0E.2B.34.01.01.01.01. 0E.01.01.01.1E.00.00.00 (CRC 40558)	float	4 or 8 (see (1) below)	No

(1) In this pack the three elements must use the same length; their individual lengths are computed using the total length of the pack divided by three, as shown in the equation below.

$$Length_{Float} = Length_{Pack}/3$$

Requirement	
ST 1504-13	A Node Vector Pack shall contain Unit Vectors of the same length.

7.8 Specific Angular Momentum Vector Pack

Table 10 defines the elements in the Specific Angular Momentum Vector Pack. The *Pack Key* and *Name* at the top of the table uniquely identify the pack, and are registered in the KLV Dictionary. The *Name* and *Key* columns are the identification of the elements. The *Type* column defines the format of the data for the given element. The *Units* column indicates the units used for the measurement of the element. The *Length* column defines the number of bytes an element uses in the pack. The *SDCC-FLP Source List* indicates if the element is part of the Specific Angular Momentum Vector Pack's Source List.

This pack is optional, but when used, all elements in the pack are mandatory.

The Specific Angular Momentum Vector is discussed in Section 6.

Table 10: Specific Angular Momentum Vector Pack

Pack Key					Name
06.0E.2B.34.02.05.01.01.0E.01.03.01.0C.05.00.00 (CRC 23753)					Specific Angular Momentum Pack
Constituent Elements					
Name	**Key**	**Format**	**Units**	**Length**	**SDCC-FLP Source List**
SAM Vector X Component	06.0E.2B.34.01.01.01.01.0E.01.01.01.27.00.00.00 (CRC 16880)	float	m^2/s	4 or 8 (see (1) below)	No
SAM Vector Y Component	06.0E.2B.34.01.01.01.01.0E.01.01.01.28.00.00.00 (CRC 38174)	float	m^2/s	4 or 8 (see (1) below)	No
SAM Vector Z Component	06.0E.2B.34.01.01.01.01.0E.01.01.01.29.00.00.00 (CRC 58282)	float	m^2/s	4 or 8 (see (1) below)	No

(1) In this pack the three elements must use the same length; their individual lengths are computed using the total length of the pack divided by three, as shown in the equation below.

$$Length_{Float} = Length_{Pack}/3$$

7.9 Scalar Auxiliary Parameters

The Scalar Auxiliary Parameter (Table 11) represents the values of scalar auxiliary parameters. Definitions and usage are given in Section 6.

Table 11: Scalar Auxiliary Parameters

Parameter	Symbol	Units	Length
Mean Motion	n	$^\circ/s$	4 or 8 bytes
Mu	μ	m^3/s^2	4 or 8 bytes
Semilatus Rectum	p	M	4 or 8 bytes
Specific Energy	ζ	J/kg	4 or 8 bytes

7.10 Standard Deviation and Correlation Coefficient Floating Length Pack (SDCC-FLP)

The Standard Deviation and Correlation Coefficient Floating Length Pack (FLP) can be invoked to provide standard deviations for the individual elements, and also provide correlation coefficients between any elements in the Source List. The Source List for this Local Set is defined in the *SDCC-FLP Source List* column of Table 1, along with embedded packs of this Local Set.

In applying the SDCC-FLP Tag, it is advised to review the usage of the SDCC-FLP (Standard Deviation Correlation Coefficient Floating Length Pack) construct presented in MISB ST 1010 [7].

The SDCC defines a compact structure for two data lists: Standard Deviation and Cross Correlation values. The data type and size for each list must be self-consistent; all Standard Deviation values must be the same type and size; all Cross Correlation values must be the same type and size. The type and size of each list can be determined at runtime.

Important: In version 0 of ST1603 the Standard Deviation values are restricted to IEEE floating point values. Future versions of ST 1603 may allow for the use IMAP values after appropriate limits are defined for each Standard Deviation.

Cross Correlation, values may use either IEEE or IMAP types as needed by the system producing the SDCC pack. Each value indicated with a “Y” in the SDCC FLP column of Table 1 can have uncertainty (i.e. standard deviation or sigma, σ) computed or measured information. Additionally, each value can be cross correlated to any of the other value resulting in a potential correlation coefficient value for that pair of values. Values with no correlation result in a correlation coefficient value of zero for that pair of values.

MISB ST 1010 defines how to package the standard deviation and correlation coefficient values. Per ST 1010, at runtime the list of values with standard deviation values defined constitutes the Refined Source List. The Refined Source List values are written into the State Vector Local Set immediately followed by the SDCC-FLP, where each row of the SDCC-FLP upper triangular matrix is in the same order as the values just written in the Local Set.

The SDCC-FLP has five defining parameters: Matrix Size, Parse Control, Bit Vector, Standard Deviation Elements (values), and the Correlation Coefficient Elements (values).

7.10.1 Matrix Size

The Matrix Size is the length of the Refined Source List. This value will be less than or equal to the size of the Source List.

7.10.2 Parse Control

Mode 2 Parse Control is the mode used for MISB ST 1010. Consult MISB ST 1010 for further description of Mode 1 and 2 of the Parse Control.

Requirement	
ST 1504-14	The State Vector Local Set shall only include SDCC-FLPs using Mode 2 Parse Control, as defined in MISB ST 1010.

Five values in the Mode 2 Parse Control are computed at runtime: Cs, S_f, S_{len}, C_f, and C_{len}.

- The C_s value indicates if the correlation coefficient values are sparsely represented in the SDCC-FLP.
- The S_f value defines the data format type of the standard deviation values, either IMAP (see MISB ST 1201 [15]) or IEEE Floating Point values. ST 1010 does not allow the mixing of types; therefore, all standard deviation values need to be converted to one type.

- The S_{len} value defines the number of bytes used by each standard deviation value. If a system requires greater precision, more bytes can be added.
- The C_f value defines the data format type of the correlation coefficient values (i.e. either IEEE Floating Point or ST 1201 mapped values).
- The C_{len} value defines the number of bytes for each correlation coefficient value. Systems requiring greater precision can use more bytes.

7.10.3 Bit Vector

As discussed in ST 1010 correlation coefficient data can be a sparse matrix. The Bit Vector indicates where to eliminate the zeros in the SDCC-FLP. See Appendix A in ST 1010 to determine when the Bit Vector should be used. The decision to use the Bit Vector can be made at run time.

7.10.4 Standard Deviation Values

The standard deviation values in IEEE Floating Point, and included in the SDCC-FLP in the same order of the Refined Source List.

7.10.5 Correlation Coefficient Values

The correlation coefficient values converted to the desired data format, either IEEE Floating Point or ST 1201 mapped values, and included in the SDCC-FLP. The rows and columns of the correlation coefficient matrix are in the same order as the Refined Source List.

7.11 Research and Experimentation

To enable research and experimentation using the ST 1504 Local Set, the Reference Frame, State Vector Type and Anomaly Type each allow a RESERVED value of zero (0). Using the value of zero for any of these items produces a Local Set that is NOT CONFORMANT and may NOT be interoperable with other standards conformant systems. Therefore, a system using a value of zero for any of these items is only to be used within a local environment (i.e. a closed system).

Should research and experimentation demonstrate a need for a new State Vector method, this standard will be updated to reflect the new State Vector method. A new value in the RESERVED space above zero will be defined for the appropriate table type.

8 Mathematical Details (Informative)

8.1 Rotation Matrices for Epoch Conversion

The angles that specify the position of the mean equinox and equator of date with respect to the mean equator and equinox of the J2000 system are:

$$\zeta_A = 0°.6406161T + 0°.0000839T^2 + 0°.0000050T^3$$

$$z_A = 0°.6406161T + 0°.0003041T^2 + 0°.0000051T^3$$

$$\theta_A = 0°.5567530T - 0°.0001185T^2 - 0°.0000116T^3$$

where T = (JD – 2451545.0)/36525. That is to say, the difference in time from noon on 1 January 2000 in Julian Centuries. The physical meaning of these rotation angles is shown in Figure 4.

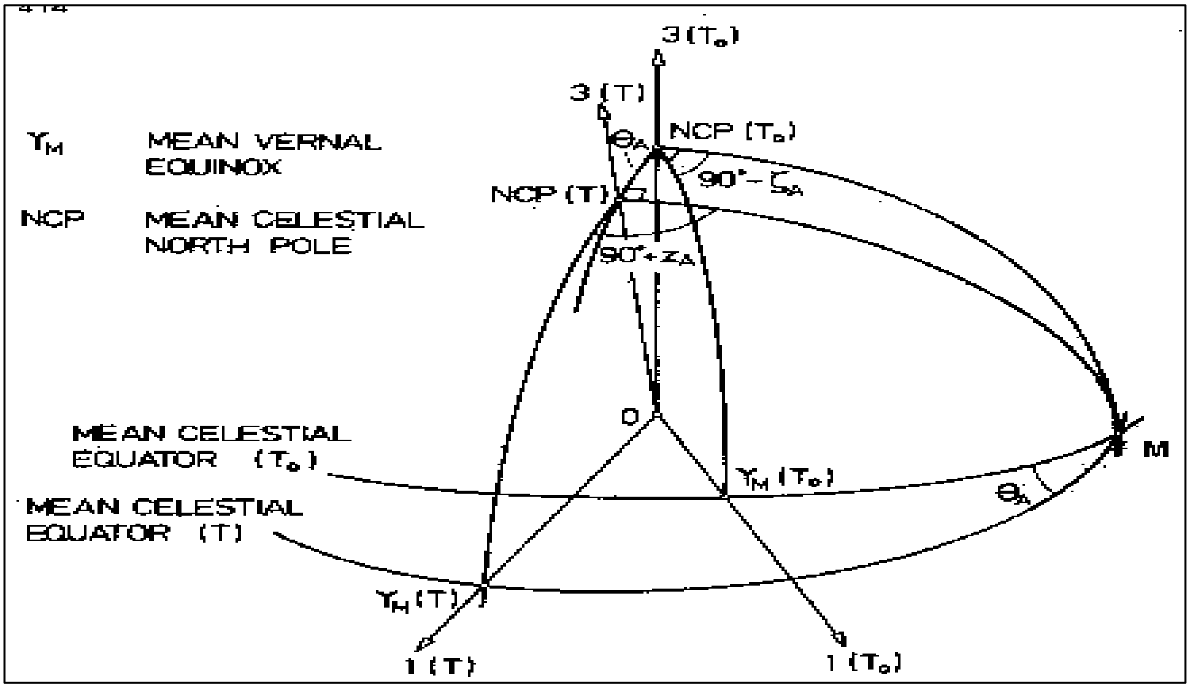

Figure 4: Rotation Angles for Coverting from J2000 to Mean of Date Frames

The rotation matrix from J2000 (inertial coordinates) to the Mean of Date (inertial coordinates) is given in the *Astronomical Almanac* (p. B18) [3] as:

$$P = \begin{bmatrix} \cos\zeta_A \cos\theta_A \cos z_A - \sin\zeta_A \sin z_A & -\sin\zeta_A \cos\theta_A \cos z_A - \cos\zeta_A \sin z_A & -\sin\theta_A \cos z_A \\ \cos\zeta_A \cos\theta_A \sin z_A + \sin\zeta_A \cos z_A & -\sin\zeta_A \cos\theta_A \sin z_A + \cos\zeta_A \cos z_A & -\sin\theta_A \sin z_A \\ \cos\zeta_A \sin\theta_A & -\sin\zeta_A \sin\theta_A & \cos\theta_A \end{bmatrix}$$

which is equivalent to $\mathbf{P} = \mathbf{R_z}(-z_A)\mathbf{R_y}(\theta_A)\mathbf{R_z}(-\zeta_A)$ (a rotation about the z-axis by $-\zeta_A$, followed by a rotation about the y-axis by θ_A, and a final rotation about the z-axis by $-z_A$).

Another rotation matrix may be applied to the Mean-of-Date frame to take into account the effects of nutation on the pole and the equator. This will yield another (and more accurate) inertial coordinate system at a particular epoch (time), called the True-of-Date frame.

The Mean Obliquity of Date is given by

$$\varepsilon_o = \varepsilon_{o*} - 46''.815T - 0''.0006T^2 + 0''.00181T^3,$$

where T is again in Julian Centuries from the J2000 epoch and

$$\varepsilon_{o*} = 23° \ 26' \ 21.448.$$

The True Obliquity is given by $\varepsilon = \varepsilon_o + \Delta\varepsilon$ as shown in Figure 5. The rotation matrix from Mean of Date to True of Date (inertial coordinates) is taken from the *Explanatory Supplement to the Astronomical Almanac* (p. 114) [8].

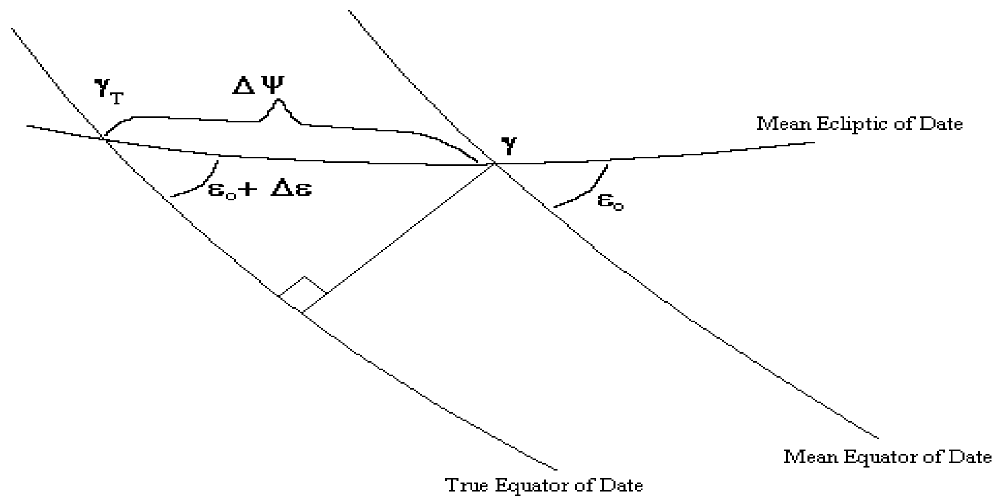

Figure 5: True Obliquity

The rotation matrix from Mean-of-Date to True-of-Date is given by $\mathbf{N} = \mathbf{R}_x(-\varepsilon)\mathbf{R}_z(-\Delta\psi)\mathbf{R}_y(\varepsilon_0)$, with the angles as shown above. When fully expanded, the rotation matrix takes the form:

$$N = \begin{bmatrix} \cos\Delta\psi & -\sin\Delta\psi\cos\varepsilon_0 & -\sin\Delta\psi\sin\varepsilon_0 \\ \sin\Delta\psi\cos\varepsilon & \cos\Delta\psi\cos\varepsilon\cos\varepsilon_0 + \sin\varepsilon\sin\varepsilon_0 & \cos\Delta\psi\cos\varepsilon\sin\varepsilon_0 - \sin\varepsilon\cos\varepsilon_0 \\ \sin\Delta\psi\sin\varepsilon & \cos\Delta\psi\sin\varepsilon\cos\varepsilon_0 - \cos\varepsilon\sin\varepsilon_0 & \cos\Delta\psi\sin\varepsilon\sin\varepsilon_0 + \cos\varepsilon\cos\varepsilon_0 \end{bmatrix}$$

The combined rotation matrix from J2000 to True of Date is therefore given by $\mathbf{Q} = \mathbf{NP}$.

8.2 Position and Velocity to Keplerian Elements

The instantaneous displacement and velocity vectors of the satellite are given by:

$$\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$$
$$\vec{v} = (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}),$$

from which it follows that

$$|\vec{r}| = \sqrt{(x^2 + y^2 + z^2)}$$
$$|\vec{v}| = \sqrt{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}.$$

The specific angular momentum vector of the satellite is

$$\vec{h} = \vec{r} \times \vec{v}.$$

From this, the inclination of the satellite is computed, since the inclination is also the angle between the angular momentum vector and the z-axis:

$$i = \arccos\left(\frac{h_z}{|\vec{h}|}\right).$$

The angular momentum vector is used to compute Ω. Consider the projection of $\vec{h}$ onto the x-y plane. The following two relationships must hold:

$$\sin\Omega = \frac{h_x}{\sqrt{h_x^2 + h_y^2}} \text{ and } \cos\Omega = -\frac{h_y}{\sqrt{h_x^2 + h_y^2}},$$

from which Ω is uniquely defined.

The specific energy (energy per unit mass) of the satellite at any point along its orbit is given by

$$\xi = \frac{|\vec{v}|^2}{2} - \frac{\mu}{|\vec{r}|},$$

where $\mu = G(M_{Earth} + M_{Satellite})$. The first term in the energy equation is the kinetic energy contribution and the second is the potential energy contribution to the specific energy. The negative sign forces the potential energy to be negative, in keeping with the convention that an object at rest infinitely far away from the central body has zero energy.

From the energy equation, both the semi-major axis and the eccentricity of the orbit can be determined:

$$a = -\frac{\mu}{2\xi} \text{ and } e = \left[1 + \frac{2\xi|\vec{h}|^2}{\mu^2}\right]^{1/2}.$$

The True Anomaly satisfies the two conditions

$$\cos\nu = \frac{1}{|\vec{r}|e}\left[\frac{|\vec{h}|^2}{\mu} - |\vec{r}|\right] \text{ and } \sin\nu = \frac{|\vec{h}|}{\mu e}\frac{\vec{r}\cdot\vec{v}}{|\vec{r}|}.$$

These two expressions may be simplified somewhat by using the *semilatus rectum*, an amazingly convenient auxiliary parameter of the ellipse given by

$$p = \frac{|\vec{h}|^2}{\mu}.$$

There are many other ways to express the *semilatus rectum*, such as $p = a(1-e^2)$, but this is the most physically meaningful formulation for the Two Body Problem.

The last orbital element to compute is the argument of perigee, ω. This is obtained from the expressions

$$\cos(\omega+\nu)=\frac{x}{|\vec{r}|}\cos\Omega+\frac{y}{|\vec{r}|}\sin\Omega \text{ and } \sin(\omega+\nu)=\frac{z}{|\vec{r}|\sin i}.$$

Since the True Anomaly, ν, was previously determined, the expression (ω+ν) for the Argument of Perigee is readily solved. The expression (ω+ν) is referred to as the Argument of Latitude, and sometimes denoted by *u*.

There are other ways of determining the orbital elements from the state, some of which are very computationally efficient for computer coding. The oft-rediscovered eccentricity vector points towards the periapsis of the orbit, and has magnitude equal to the eccentricity of the orbit. It is given by:

$$\vec{e}=\frac{\vec{v}\times\vec{h}}{\mu}-\frac{\vec{r}}{|\vec{r}|}.$$

The node vector points towards the ascending node and is constructed as follows:

$$\vec{n}=-h_y\hat{i}+h_x\hat{j}.$$

From these, it follows that

$$\cos\omega=\frac{\vec{e}\cdot\vec{n}}{e|\vec{n}|}.$$

A quadrant check will be needed to ensure the correct angle is chosen. The True Anomaly can be determined in a similar fashion via

$$\cos\nu=\frac{\vec{e}\cdot\vec{r}}{e|\vec{r}|},$$

again with a quadrant check being necessary.

From the true anomaly, the eccentric anomaly, E, is determined via the equation

$$\cos E=\frac{e+\cos\nu}{1+e\cos\nu}.$$

The eccentric anomaly must be in the same half-plane as the true anomaly. The mean anomaly then follows (trivially, going this way) from Kepler's Equation:

$$M=E-e\sin E.$$

The mean anomaly is also given by

$$M=M_0+nt=M_0+\sqrt{\frac{\mu}{a^3}}t.$$

The mean anomaly is not just an angle; it is the parameter measuring Kepler's Second Law: *The line connecting the primary to the orbiting body sweeps out equal area in equal time.*

8.3 Keplerian Elements to Position and Velocity

Of course, it is possible to reconstruct the position and velocity of a satellite at any given time from the orbital elements at that time. The derivation is a straightforward but extremely messy exercise in spherical geometry, so only the results are presented. First, define some mathematical shorthand:

$$v_r = \dot{r} = \frac{he}{p}\sin\nu$$
$$v_\theta = \frac{h}{r}$$
$$\dot{x}^* = v_r \cos\nu - v_\theta \sin\nu$$
$$\dot{y}^* = v_r \sin\nu + v_\theta \cos\nu$$
$$Q_{11} = \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i$$
$$Q_{12} = -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i$$
$$Q_{21} = \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i$$
$$Q_{22} = -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i$$
$$Q_{31} = \sin\omega\sin i$$
$$Q_{32} = \cos\omega\sin i\,.$$

With these terms so defined, the position and velocity components are given by:

$$x = r\cos\nu Q_{11} + r\sin\nu Q_{12}$$
$$y = r\cos\nu Q_{21} + r\sin\nu Q_{22}$$
$$z = r\cos\nu Q_{31} + r\sin\nu Q_{32}$$
$$\dot{x} = \dot{x}^* Q_{11} + \dot{y}^* Q_{12}$$
$$\dot{y} = \dot{x}^* Q_{21} + \dot{y}^* Q_{22}$$
$$\dot{z} = \dot{x}^* Q_{31} + \dot{y}^* Q_{32}\,.$$

8.4 Keplerian Elements to Non-Singular Equinoctial Elements

To avoid the singularities that may result from use of Keplerian elements, the Keplerian elements may be converted to Non-Singular Equinoctial elements, as follows:

a Semi-major Axis

$$\left.\begin{aligned} h &= e\sin(\omega + \Omega) \\ k &= e\cos(\omega + \Omega) \end{aligned}\right\} \text{ Components of Eccentricity Vector}$$

$$\left.\begin{aligned} p &= \tan(\tfrac{i}{2})\sin\Omega \\ q &= \tan(\tfrac{i}{2})\cos\Omega \end{aligned}\right\} \text{ Components of Node Vector}$$

$\lambda = M + \omega + \Omega$ Mean Longitude

8.5 Position and Velocity to Non-Singular Equinoctial Elements

This section shows how to convert from Cartesian coordinates to non-singular equinoctial elements.

The semi-major axis is derived from the specific energy:

$$a = \frac{1}{\frac{2}{|\vec{r}|} - \frac{|\vec{v}|^2}{\mu}}$$

where $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ and $\vec{v} = (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k})$.

Now define

$$\hat{w} = \frac{\vec{r} \times \vec{v}}{|\vec{r} \times \vec{v}|}.$$

From this,

$$p = \frac{w_x}{1+w_z} \text{ and } q = -\frac{w_y}{1+w_z}.$$

Now, let

$$\hat{f} = \frac{1}{1+p^2+q^2}\begin{bmatrix} 1-p^2+q^2 \\ 2pq \\ -2Ip \end{bmatrix}$$

$$\hat{g} = \frac{1}{1+p^2+q^2}\begin{bmatrix} 2p \\ I(1+p^2-q^2) \\ 2q \end{bmatrix}$$

Where $\hat{f}, \hat{g},$ and $\hat{w}$ form the basis vectors of the equinoctial reference frame.

Now, using the eccentricity vector

$$\vec{e} = \frac{\vec{v} \times \vec{h}}{\mu} - \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{v} \times (\vec{r} \times \vec{v})}{\mu} - \frac{\vec{r}}{|\vec{r}|}$$

gives,

$$h = \vec{e} \cdot \hat{g} \text{ and}$$

$$k = \vec{e} \cdot \hat{f}.$$

The mean longitude can be computed from the position of the satellite in the equinoctial reference frame:

$$X = \vec{r} \cdot \hat{f} \text{ and } Y = \vec{r} \cdot \hat{g}.$$

Let

$$b = \frac{1}{1+\sqrt{1-h^2-k^2}},$$

then

$$\sin F = h + \frac{(1 - h^2 b)Y - hkbX}{a\sqrt{1 - h^2 - k^2}}$$

and

$$\cos F = k + \frac{(1-k^2 b)X - hkbY}{a\sqrt{1-h^2-k^2}}.$$

Once the sine and cosine of the eccentric longitude are known, the mean longitude comes from Kepler's Equation:

$$\lambda = F + h\cos F - k\sin F.$$

8.6 Non-singular Equinoctial Elements to Position and Velocity

To go from equinoctial elements back to Cartesian position and velocity, recall that

$$\lambda = M + \omega + \Omega,$$

eccentric longitude $F = E + \omega + \Omega$, and

$$L = \nu + \omega + \Omega.$$

From which it follows that

$$\lambda = F + h\cos F - k\sin F \text{ and}$$

$$r = a[1 - h\sin F - k\cos F].$$

Note that the F cannot be solved for in closed form going this way. Once an iterative solution has been found, the position vector is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1+p^2+q^2} \begin{bmatrix} 1 - p^2 + q^2 & 2pq & 2p \\ 2pq & 1 + p^2 - q^2 & -2q \\ -2p & 2q & I(1 - p^2 - q^2) \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ 0 \end{bmatrix},$$

where

$$X_1 = r\cos L = a[\cos F - k - \frac{h(\lambda - F)}{1 + \sqrt{1 - e^2}}]$$

and

$$Y_1 = r\sin L = a[\sin F - h - \frac{k(\lambda - F)}{1 + \sqrt{1 - e^2}}].$$

The velocity components can be derived by variation of parameters. See [9] for details. The results are given in [1].

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = n \begin{bmatrix} \frac{x}{|r|}\frac{d|r|}{d\lambda} + \sigma\left[(1 + q^2 - p^2)\frac{d\cos L}{dF} + 2pq\frac{d\sin L}{dF}\right] \\ \frac{y}{|r|}\frac{d|r|}{d\lambda} + \sigma\left[(1 + p^2 - q^2)\frac{d\sin L}{dF} + 2pq\frac{d\cos L}{dF}\right] \\ \frac{z}{|r|}\frac{d|r|}{d\lambda} + 2\sigma\left[q\frac{d\sin L}{dF} - p\frac{d\cos L}{dF}\right] \end{bmatrix},$$

where

$$|r| = a(h\sin F - k\cos F)$$

and

$$\sigma = \frac{a}{1+p^2+q^2}.$$